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FUZZY DIFFERENTIAL INCLUSIONS*

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Differential equations are considered with some unknown parameters such that there is, however, some information available about certain preferred values of the parameters. Using a formal approach to such information, in accordance with the theory of fuzzy sets (FS), we introduce the notation of a differential inclusion (DI) with a fuzzy right-hand side. The solution of a differential inclusion is defined as the FS of motions. It is established that the level set of this FS is identical with the bundle of solutions of an ordinary differential equation whose right-hand side is given by the corresponding level set of the fuzzy right-hand side. Conditions for the right-hand side of the original differential equation and for the membership function of the FS of parameters are stated which ensure that there exists a solution of the DI with a fuzzy right-hand side. A special case of a controlled linear system is considered with a matrix of coefficients defined by means of the direct product of one-dimensional FS's. In this way a new formalism for differential systems with fuzzy unknown parameters is proposed. A connection between the theory of DI's and FS's is established.

The origin of DI's with a fuzzy right-hand side can be illustrated by the following example. Suppose that we have a differential equation which models a real process

$$x^{\cdot} = f(t, x, k) \tag{0.1}$$

where k is a vector formed by the parameters on the right-hand side of the equation. The vector k may often be completely unknown. Moreover, k may vary according to an unknown law. If some set K of possible values of k can be defined, then it is convenient to replace (0.1) by the DI

$$x^{\cdot} \in f(t, x, K) \tag{0.2}$$

It may happen that different points of K do not have an equal status as possible samples of the values of k. Then, it is natural to regard K as a FS. If the function $f(t, x, \cdot)$ is extended onto the family of FS's in accordance with Zadeh's generalization principle /1/, we obtain a FS on the right-hand side of (0.2).

1. Differential inclusions with a fuzzy right-hand side. We introduce the following notation: $P(X)$ is the set of fuzzy subsets of the space X /1, 2/, μ_M is the membership function for a FS M, and $\langle \mu_M, x\rangle$ is the value of μ_M at a point x.

Let an interval $T \stackrel{\Delta}{=} [t_*, +\infty)$, a domain $G \subset T \times R^n$, and a function

$$\Psi : G \to P(R^n) \tag{1.1}$$

be given.

Definition 1. We define a DI with a fuzzy right-hand side to be the expression

$$x^{\cdot} \in \Psi(t, x) \tag{1.2}$$

representing a formal symbolic relation whose essential meaning is defined by the FS of solutions of (1.2).

Let $[t_*, t^*] \subset T$ and let $\mathrm{AC}[t_*, t^*]$ be the set of absolutely continuous functions on $[t_*, t^*]$, whose graph is contained in G.

Definition 2. The FS $R[t_*, t^*]$ in $\mathrm{AC}[t_*, t^*]$ whose membership function is defined by the relation

$$\langle \mu_{R[t_*, t^*]}, x(\cdot)\rangle \stackrel{\Delta}{=} \operatorname*{essinf}_{t\in[t_*, t^*]} \langle \mu_{\Psi(t, x(t))}, x^{\cdot}(t)\rangle \tag{1.3}$$

will be called the FS of solutions of (1.2) in the interval $[t_*, t^*]$.

From the point of view of the theory of fuzzy sets, the sense of the definition can be,

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essentially, reduced to the following. Let a family of spaces X_t with $t \in [t_*, t^*]$ be given and let M_t be a FS in X_t. The elements of the direct product $X \triangleq \prod_t X_t$ are functions $x(\cdot)$ that satisfy the condition $x(t) \in X_t$. The direct product $M \triangleq \prod_t M_t$ of FS's M_t is defined by the membership function

$$\langle \mu_M, x(\cdot)\rangle = \inf_t \langle \mu_{M_t}, x(t)\rangle$$

In the case in question $X_t = R^n$, $M_t = \Psi(t, x(t))$, and the membership degree of $x(\cdot)$ in the set of solutions of inclusion (1.2) is set to be equal to the membership degree of the derivative $\dot{x}(\cdot)$ in the direct product $\prod_t M_t$ to within a set of measure zero of values of t.

For a FS M in X and for $a \in [0, 1]$, the ordinary (non-fuzzy) set defined by the relation

$$M_a \triangleq \{x \in X : \langle \mu_M, x\rangle \geqslant a\}$$

is called the level set M_a.

We denote the level sets for the FS's $\Psi(t, x)$ and $R[t_*, t^*]$ by $\Psi_a(t, x)$ and $R_a[t_*, t^*]$.

Proposition 1. For the condition

$$x(\cdot) \in R_a[t_*, t^*] \tag{1.4}$$

to be satisfied it is necessary and sufficient that $x(\cdot)$ be a solution of the ordinary DI

$$\dot{x}(t) \in \Psi_a(t, x(t)) \tag{1.5}$$

Proof. Let condition (1.4) be satisfied. then

$$\operatorname{essinf}_t \langle \mu_{\Psi(t, x(t))}, \dot{x}(t)\rangle \geqslant a \tag{1.6}$$

We denote by T_0 the set of all t such that either $\langle \mu_{\Psi(t, x(t))}, \dot{x}(t)\rangle < a$ or the function $\dot{x}(t)$ is not defined. The set T_0 is a set of measure zero. Therefore,

$$\langle \mu_{\Psi(t, x(t))}, \dot{x}(t)\rangle \geqslant a \tag{1.7}$$

almost everywhere. This means that (1.5) holds almost everywhere.

Conversely, let (1.5) be satisfied for almost all t. Thus, (1.7) holds for almost all t. Hence, it follows that inequality (1.6) holds or $\langle \mu_{R[t_*, t^*]}, x(\cdot)\rangle \geqslant a$. Therefore, condition (1.4) is satisfied.

It follows that the level sets of the FS of solutions of the DI (1.2) are the sets of solutions of the ordinary DI (1.5) with $a \in [0, 1]$. In this connection, it becomes an important question whether the sets $\Psi_a(t, x)$ are non-empty and compact and whether the multi-valued functions $\Psi_a(t, x)$ are convex and semicontinuous from above.

Let us investigate the conditions under which the sets $\Psi_a(t, x)$ are non-empty for all $(t, x) \in G$. A FS is said to be empty $(\varnothing)$, if its membership function vanishes identically. We assume that $\Psi(t, x) \neq \varnothing$ for all $(t, x) \in G$.

Definition 3. A FS A is said to be regular if it is non-empty, its level sets A_a are compact for $a > 0$, and the membership function μ_A is semicontinuous from above.

For a regular FS A, there exists the $\max\{a: A_a \neq \varnothing\}$.

Indeed, let $A_{a'} \neq \varnothing$. Since μ_A is semicontinuous from above and $A_{a'}$ is compact, the function μ_A reaches a maximum on $A_{a'}$ at some point x^*. Let $a^* \triangleq \langle \mu_A, x^*\rangle$. We assert that $a^* = \max\{a: A_a \neq \varnothing\}$. We have $A_{a^*} \neq \varnothing$, since $x^* \in A_{a^*}$. If $a > a^*$, then $A_a = \varnothing$. Otherwise, $\langle \mu_A, x\rangle \geqslant a > a^*$ for some $x \in A_{a'}$, which contradicts the definition of a^*.

Let

$$\zeta(t, x) \triangleq \max\{a : \Psi_a(t, x) \neq \varnothing\}$$

We remark that $\zeta(t, x) > 0$ by virtue of $\Psi(t, x) \neq \varnothing$. We set $a_* = \inf_G \zeta(t, x)$. Then for all $(t, x) \in G$ and $a \leqslant a_*$ we have $\Psi_a(t, x) \neq \varnothing$. But if $a > a_*$, then there is a point $(t_0, x_0) \in G$ such that $\zeta(t_0, x_0) < a$, and so $\Psi_a(t_0, x_0) = \varnothing$.

In what follows we shall assume that $a_* > 0$, and the level sets $\Psi_a(t, x)$ will be considered for $a \in [0, a_*]$. The case $\zeta(t, x) \equiv \text{const} = a_*$ and, in particular, the case $a_* = 1$ should be considered separately.

A FS A is said to be convex if each of its level sets is convex (/1/, p.161). If A^α is a family of FS's and $A \triangleq \bigcap_\alpha A^\alpha$, then $A_a = \bigcap_\alpha A_a^\alpha$ /1/. It follows that the intersection of any family of convex FS's is a convex FS. Let $\operatorname{co} A$ be an intersection of all convex FS's containing A (the convex hull of A). Then $(\operatorname{co} A)_a = \operatorname{co}(A_a)$.

The following criterion can be used to establish the semicontinuity from above of the functions $\Psi_a(t, x)$

Proposition 2. Let $\zeta(t, x) = \mathrm{const} = a_*$. In order that the multivalued function $\Psi_a(t, x)$ be semicontinuous from above with respect to (t, x) it is necessary and sufficient that

$\nu \triangleq \langle \mu_{\Psi(t,x)}, y\rangle$ be semicontinuous from above as a function of (t, x, y).

Proof. Sufficiency. Suppose that ν is semicontinuous from above must $\Psi_a(t, x)$ is not. Then the graph of Ψ_a is not closed and there are sequences $(t_n, x_n) \to (t_0, x_0)$, $y_n \to y_0$ such that $y_n \in \Psi_a(t_n, x_n)$ and $y_0 \notin \Psi_a(t_0, x_0)$. Therefore, $\nu_n \geqslant a$ and $\nu_0 < a$, where $\nu_m = \langle \mu_{\Psi(t_m, x_m)}, y_m\rangle$.
Since ν is semicontinuous from above for all $\varepsilon > 0$, the inequalities $\nu_n < \nu_0 + \varepsilon$ hold starting from some n. We have $\nu_0 < a \leqslant \nu_n < \nu_0 + \varepsilon$. Letting ε tend to zero, we obtain $\nu_0 < \nu_0$.

Necessity. Suppose that ν is not semicontinuous from above at (t_0, x_0, y_0). Then there is an $\varepsilon > 0$ and a sequence $(t_n, x_n, y_n) \to (t_0, x_0, y_0)$ such that $\nu_n \geqslant \nu_0 + \varepsilon$. Let $\nu_0 = a$. Then $y_n \in \Psi_{a+\varepsilon}(t_n, x_n)$. Therefore, $y_0 \in \Psi_{a+\varepsilon}(t_0, x_0)$ and we arrive at a contradiction: $a \geqslant a + \varepsilon$.

2. *The DI with a fuzzy right-hand side generated by a differential equation.* Let a differential equation

$$x^{\cdot} = f(t, x, k(t)),\ (t, x) \in G,\ k(t) \in R^r \tag{2.1}$$

be given ($k(t)$ is a vector of parameters (coefficients)).
Using Zadeh's generalization principle /1/, we extend the function $f(t, x, \cdot) : k \to f(t, x, k)$ to $f(t, x, \cdot) : P(R^r) \to P(R^n)$. For $K \subset P(R^r)$, the membership function of the FS $f(t, x, K)$ is defined by the equality

$$\langle \mu_{f(t,x,K)}, y\rangle = \sup_{k\in R^r} \{\langle \mu_K, k\rangle \wedge \langle \mu_{f(t,x,k)}, y\rangle\} = \sup\{\langle \mu_K, k\rangle : f(t, x, k) = y\}$$

Here and below $\wedge$ denotes the operation of taking the minimum. Moreover, it is assumed that $\sup \varnothing = 0$ since the values of $\langle \mu_K, k\rangle$ belong to $[0, 1]$.
Let $T_* = [t_*, t^*)$, and if $(t, x) \in G$ then $t \in T_*$. t^* may be equal to $+\infty$.
We define a function $K(t)$ on T_* with values in $P(R^r)$, and for Eq.(2.1) we write down the DI with a fuzzy right-hand side

$$x^{\cdot} \in f(t, x, K(t)) \tag{2.2}$$

i.e., here $\Psi(t, x) \triangleq f(t, x, K(t))$.

Proposition 3. Let $f(t, x, k)$ be a continuous function with respect to k and let $K(t)$ be a regular FS. Then for $a > 0$,

$$f_a(t, x, K(t)) = f(t, x, K_a(t))$$

Proof. Let $a > 0$ and $y \in f_a(t, x, K(t))$. We have $a' \triangleq \langle \mu_{f(t,x,K(t))}, y\rangle = \sup\{\langle \mu_{K(t)}, k\rangle : f(t, x, k) = y\} \geqslant a$. We set

$$M \triangleq \{k : \langle \mu_{K(t)}, k\rangle \geqslant a'', f(t, x, k) = y\} = \\ \{k : k \in K_{a''}(t), f(t, x, k) = y\}, 0 < a'' < a$$

It follows from the inequality $a'' < a'$ that

$$\sup\{\langle \mu_{K(t)}, k\rangle : k \in M\} = a'$$

Since $\mu_{K(t)}$ is semicontinuous from above, there is an element $k' \in M$, such that $a' = \langle \mu_{K(t)}, k'\rangle$. Moreover $y = f(t, x, k')$ and $k' \in K_{a'}(t) \subset K_a(t)$. Therefore, $y \in f(t, x, K_a(t))$.
Conversely, let $y \in f(t, x, K_a(t))$. Then $y = f(t, x, k')$, where $k' \in K_a(t)$. Thus,

$$\langle \mu_{f(t,x,K(t))}, y\rangle = \sup\ \{\langle \mu_{K(t)}, k\rangle : \\ f(t, x, k) = y\} \geqslant \langle \mu_{K(t)}, k'\rangle \geqslant a$$

It follows that $y \in f_a(t, x, K(t))$.

Corollaries. 1°. Let $f(t, x, k)$ be a continuous function with respect to k and let $K(t)$ be a regular FS. Then for $a > 0$, the level sets $f_a(t, x, K(t))$ are compact.
2°. Let $f(t, x, k)$ be a continuous function with respect to k and let $K(t)$ be a rather FS for all t. Then $\zeta(t, x) = \max\{a\colon K_a(t) \neq \varnothing\}$.

Proposition 4. Let $f(t, x, k)$ be continuous with respect to (t, x, k) (with respect to (x, k)), let $K(t)$ be a regular FS for all t, and let $K_a(t)$ be a semicontinuous function from above with respect to t for $a \in (0, a_*]$. Then for $a \in (0, a_*]$, the function $f_a(t, x, K(t))$ is semicontinuous from above with respect to (t, x) (semicontinuous from above with respect to x).

We shall give the proof in the case when continuity with respect to (t, x, k) is assumed. In the case when continuity with respect to (x, k) is assumed, the proof is analogous.

Suppose that the converse statement holds. Then we can find an open set $B \subset R^n$ such that

$$H \triangleq \{(t, x) : f_a(t, x, K(t)) \subset B\}$$

is not open. Thus, there is $(t_0, x_0) \in H, (t_n, x_n) \in H\ (n = 1, 2, \ldots)$ such that $(t_n, x_n) \to (t_0, x_0)$. We choose a sequence

$$y_n \in f_a(t_n, x_n, K(t_n)) \setminus B\ (n = 1, 2, \ldots)$$

Let $k^n \in K_a(t_n)$ be such that $y_n = f(t_n, x_n, k^n)$. Since $K_a(t)$ has compact values and it is semicontinuous from above, the image of a compact set $\{t_n, t_0\}_{n=1}^{\infty}$ under this mapping is compact (/3/, p.133). Thus, we can assume without loss of generality that the sequence k^n converges to some $k^0 \in K_a(t_0)$. Therefore, there exists $y_0 \triangleq \lim y_n = f(t_0, x_0, k^0)$. Since $y_n \notin B$, we have $y_0 \notin B$. On the other hand, $y_0 \in f(t_0, x_0, K_a(t_0)) \subset B$. We have obtained a contradiction.

Corollary. If $f(t, x, k)$ is continuous with respect to (t, x, k), $K(t) \equiv K$ and K is a regular FS, then $f_a(t, x, K)$ is semicontinuous from above with respect to (t, x) for $a \in (0, a_*]$.

$K(t)$ defined as a direct product of FS's containing the parameters of the differential Eq.(2.1) is an important special case. Let

$$k(t) \triangleq (k_1(t), \ldots, k_r(t)), \quad K(t) \triangleq \prod_{i=1}^{r} K^i(t)$$

Then $\langle \mu_{K(t)}, k\rangle = \langle \mu_{K^1(t)}, k_1\rangle \wedge \cdots \wedge \langle \mu_{K^r(t)}, k_r\rangle$

$$(K_a(t) = \prod_{i=1}^{r} K_a{}^i(t)\ \text{(/1/, p.28)}).$$

It follows that if the sets $K^i(t)$ are regular, then $K(t)$ is regular.

If for $a \in (0, a_*]$ the level sets $K_a{}^i(t)$ are semicontinuous from above with respect to t, then $K_a(t)$ is semicontinuous from above with respect to t.

Indeed, suppose that this is not the case. Then, there is an open set $B \subset R^r$ such that the set $T_0 = \{t : K_a(t) \subset B\}$ is not open, i.e., there are $t_0 \in T_0$ and $t_n \notin T_0$ such that $t_n \to t_0$. Let $k^n \in K_a(t_n) \setminus B$ and $k^n = (k_1{}^n, \ldots, k_r{}^n)$ where $k_i{}^n \in K_a{}^i(t_n)$. Since the functions $K_a{}^i(t)$ have compact values and they are semicontinuous from above, we can assume that $k_i{}^n \to k_i{}^0 \in K_a{}^i(t_0)$. Let $k^\circ = (k_1{}^\circ, \ldots, k_r{}^\circ)$. Then $k^n \to k^\circ$. Since $k^n \notin B$, it follows that $k^\circ \notin B$. On the other hand,

$$k^\circ \in \prod_{i=1}^{r} K_a{}^i(t_0) = K_a(t_0)$$

Therefore, $k^\circ \in B$.

Proposition 5. Let $f(t, x, k)$ be a continuous function with respect to t, x, k, let $K(t)$ be a regular FS, and for $a \in (0, a_*]$ let the functions $K_a(t)$ be semicontinuous from above with respect to t. Moreover, let $f(t, x\ K_a(t))$ be convex sets for all $a \in (0, a_*]$. In this case, if $(t_0, x_0) \in G$ and $a \in (0, a_*]$, then there is a $d > 0$ such that the set $R_a[t_0, t_0 + d]$ of solutions of the DI $x^{\cdot} \in f_a(t, x, K(t))$ that satisfy the initial condition $x(t_0) = x_0$ is non-empty.

It follows from the assumptions of Proposition 5 and from the preceding assertions that the right-hand side $f_a(t, x, K(t))$ satisfies all the assumptions of Theorem 1 of /4/ (pp. 60-61). The proof reduces to the application of this theorem. If a domain G contains the cylinder

$$Z = \{(t, x) : t_0 \leqslant t \leqslant t_0 + c, \ \ |x - x_0| \leqslant b\}$$

then we can choose as d

$$d = \min\{c, b/m\}, \quad m = \sup_{(t, x) \in Z} |f_a(t, x, K(t))|$$

If $K(t)$ is independent of t, then the requirement that $K(t)$ should be semicontinuous from above can be omitted in the formulation of Proposition 5.

As an example, consider the linear differential equation with constant coefficients

$$x^{\cdot} = \mathbf{A}x + \mathbf{B}u(t),\ x \in R^n,\ u(t) \in R^s \tag{2.3}$$

where $\mathbf{A}$ is an $(n \times n)$-matrix, $\mathbf{B}$ is an $(n \times s)$-matrix and $u(t)$ is a continuous function. Let $\mathbf{A} = [a_{ij}]$, $\mathbf{B} = [b_{ml}]$ and $k = [a_{11}, \ldots, a_{nn}, b_{11}, \ldots, b_{ns}]$. Let A^{ij} and B^{ml} $(i, j, m = 1, \ldots, n;\ l = 1, \ldots, s)$ be FS's given on the real axis. We denote by A the FS of matrices with the membership function

$$\langle \mu_A, \mathbf{A} \rangle = \min_{i,j} \langle \mu_{A^{ij}}, a_{ij} \rangle$$

Analgously, we define the FS B of matrices. Then, setting $K \overset{\Delta}{=} A \times B$, we obtain the DI generated by Eq.(2.3) and the set K. This inclusion can be written formally as

$$x^{\cdot} \in Ax + Bu(t)\ (f(t, x, K) = Ax + Bu(t)) \tag{2.4}$$

Let A^{ij} and B^{ml} be convex regular FS's. Then K is a regular convex FS. We consider the function $\xi(t, x)\colon R^r \to R^n$ with $r = n^2 + ns$ that transforms $(\mathbf{A}, \mathbf{B})$ into $\mathbf{A}x + \mathbf{B}u(t)$. The function is linear, hence it transforms any convex set contained in the space of coefficients R^r into a convex set in R^n. Thus, the sets

$$f_a(t, x, K) \overset{\Delta}{=} A_a x + B_a u(t)$$

are convex and Proposition 5 can be applied to the DI (2.4).

3. ***DI's whose right-hand sides are measurable with respect to t.*** We consider the differential equation

$$x^{\cdot} = f(t, x, w(t), k(t)),\ (t, x) \in G,\ w \in R^s,\ k \in R^r \tag{3.1}$$

We shall assume that $f(t, x, w, k)$ is a continuous function with respect to (t, x, w, k), and $w(t)$ is measurable with respect to t. Let a multivalued function $K(t)$ be chosen. Then we obtain the DI

$$x^{\cdot} \in f(t, x, w(t), K(t)) \tag{3.2}$$

A multivalued function $F(p),\ p \in E$ is said to be measurable with respect to p if E is a measurable set and for any closed B, the set $\{p : F(p) \cap B \neq \varnothing\}$ is measurable. It turns out that the study of DI's with fuzzy right-hand sides can be reduced to the study of DI's whose right-hand sides are measurable with respect to t.

Proposition 6. Let $K(t)$ be a regular FS and let $K_a(t)$ be semicontinuous from above for any $a \in (0, a_*]$. Then $f_a(t, x, w(t), K(t))$ is measurable with respect to t for any x and $a \in (0, a_*]$.

Proof. We set $h \overset{\Delta}{=} (t, w)$, $f_1(h, x, k) \overset{\Delta}{=} f(t, x, w, k)$ and $M \overset{\Delta}{=} \{h : B \cap f_1(h, x, K_a(t)) \neq \varnothing\}$ where B is a closed set in R^n. $h \in M$ if and only if there is $k(t) \in K_a(t)$ such that $f_1(h, x, k(t)) \in B$. Let $h_n \overset{\Delta}{=} (t_n, w_n) \in M$ converge to $h_0 \overset{\Delta}{=} (t_0, w_0)$ and let $f_1(h_n, x, k(t_n)) \in B$. Since $\{t_n, t_0\}_{n=1}^{\infty}$ is a compact set and $K_a(t)$ is semicontinuous from above and has compact values, it follows that $(\bigcup_n K(t_n)) \cup K(t_0)$ is compact. Thus, we can assume without loss of generality that $k(t_n)$ converges to $k(t_0) \in K(t_0)$. But then, $f_1(h_n, x, k(t_n)) \to f_1(h_0, x, k(t_0)) \in B$. It follows that $h_0 \in M$ and M is a closed set.

We denote by T_* the set of those t for which

$$B \cap f(t, x, w(t), K_a(t)) \neq \varnothing$$

T_* is identical with the set of those t for which $h(t) \overset{\Delta}{=} (t, w(t)) \in M$. Since $w(t)$ is a measurable function, $h(t)$ is measurable (/5/, p.82). It follows that T_* is a measurable set.

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CONTROL OF THE MOTION OF A SOLID ROTATING ABOUT ITS CENTRE OF MASS*

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Problems of controlling the spherical motion of a rotating solid when the controlling torques delivered to the body by the controls do not contain an x-component and their axes are not the principal central axes of inertia of the body are investigated. It is shown that as the transverse components of the angular velocity vector are suppressed and the orientation of the non-principal axis of inertia of the body stabilizes, there is an accompanying drop in the angular velocity of twist and, in the final analysis, the rotating body loses its gyroscopic properties. On the other hand, control of the uniform rotation of the body about a principal axis of inertia and of its orientation in inertia space excludes a marked dynamical effect. Control algorithms are porposed to guarantee stability of the relevant motions of the body when the control parameters are subject to constraints. The efficiency of these solutions is confirmed by modelling experiments.

1. Statement of the problem. We introduce three right-handed Cartesian coordinate systems, all with origin O at the centre of mass of the solid: a rigidly attached system xyz, whose axes do not coincide with the system $x_*y_*z_*$ of the principal central axes of inertia of the body, and an inertial coordinate system XYZ.

The relative positions of the xyz and $x_*y_*z_*$ bases are characterized by angles ϑ, ψ and φ (Fig.1). The representation r of a vector R in the xyz basis is expressed in terms of its representation $\mathbf{r}_*$ in the $x_*y_*z_*$ basis by the formula

$$\mathbf{r} = B\mathbf{r}_*,\ B = \{\beta_{ij}\}\ (i, j = 1, 2, 3)$$

(B is the matrix of direction cosines).

Describing the rotary motion of the solid body in the xyz basis by the dynamical Euler equations

$$J\omega^{\cdot} + \omega \times J\omega = \mathbf{M},\ \omega = \{\omega_x, \omega_y, \omega_z\} \tag{1.1}$$

we note that the inertia matrix J is related to the inertia matrix $J_* = \operatorname{diag}\{J_1, J_2, J_3\}$ in the $x_*y_*z_*$ basis by the expression

$$J = BJ_*B'$$

To fix our ideas, we shall assume that $J_1 < J_2 \leqslant J_3$ (the prime in the formula denotes transposition).

It is assumed that the motion of the body is observed in the rigidly attached coordinate system xyz; the controlling torque M in that system has the following structure: $\mathbf{M} = \{0, M_y, M_z\}$.

Let ξ and η denote fixed unit vectors in the xyz basis and the XYZ inertial space, respectively. The motion of η relative to the rigidly attached coordinate system is governed by the equation

$$\eta^{\cdot} + \omega \times \eta = 0 \tag{1.2}$$

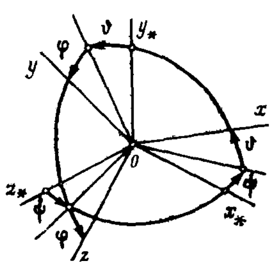

Fig.1

Assume that an angular velocity $\omega_x = \Omega$ ($|\omega_y|$, $|\omega_z| \ll \Omega$) is imported to the body.

In view of the special structure of the controlling torque M, the effect of the control

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